ESTIMATES FOR THE CONCENTRATION FUNCTIONS IN THE LITTLEWOOD–OFFORD PROBLEM

YULIA S. ELISEEVA[1], FRIEDRICH GÖTZE[2], AND ANDREI YU. ZAITSEV[1,2,3]

Abstract. Let $X, X_1, \dots, X_n$ be independent identically distributed random variables. In this paper we study the behavior of the concentration functions of the weighted sums $\sum_{k=1}^n a_k X_k$ with respect to the arithmetic structure of coefficients a_k. Such concentration results recently became important in connection with investigations about singular values of random matrices. In this paper we formulate and prove some refinements of a result of Vershynin (2011).

1. Introduction

Let $X, X_1, \dots, X_n$ be independent identically distributed (i.i.d.) random variables with common distribution $F = \mathcal{L}(X)$. The Lévy concentration function of a random variable X is defined by the equality

$$Q(F, \lambda) = \sup_{x \in \mathbf{R}} F\{[x, x+\lambda]\}, \quad \lambda > 0.$$

Let $a = (a_1, \dots, a_n) \in \mathbf{R}^n$. In this paper we study the behavior of the concentration functions of the weighted sums $S_a = \sum_{k=1}^n a_k X_k$ with respect to the arithmetic structure of coefficients a_k. Refined concentration results for these weighted sums play an important role in the study of singular values of random matrices (see, for instance, Nguyen and Vu [14], Rudelson and Vershynin [17], [18], Tao and Vu [19], [20], Vershynin [21]). In this context the problem is referred to as the Littlewood–Offord problem.

In the sequel, let F_a denote the distribution of the sum S_a, and let G be the distribution of the symmetrized random variable $\widetilde{X} = X_1 - X_2$. Let

$$M(\tau) = \tau^{-2} \int_{|x| \le \tau} x^2 \, G\{dx\} + \int_{|x| > \tau} G\{dx\} = \mathbf{E} \min\big\{\widetilde{X}^2/\tau^2, 1\big\}, \quad \tau > 0. \tag{1}$$

The symbol c will be used for absolute positive constants. Note that c can be different in different (or even in the same) formulas. We will write $A \ll B$ if $A \le cB$. Also we will write

1991 *Mathematics Subject Classification.* Primary 60F05; secondary 60E15, 60G50.

Key words and phrases. concentration functions, inequalities, the Littlewood–Offord problem, sums of independent random variables.

[1]Research supported by grant RFBR 10-01-00242.

[2]Research supported by the SFB 701 in Bielefeld.

[3]Research supported by grant RFBR 11-01-12104 and by the Program of Fundamental Researches of Russian Academy of Sciences "Modern Problems of Fundamental Mathematics".

$A \asymp B$ if $A \ll B$ and $B \ll A$. For $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ we will denote $\|x\|^2 = x_1^2 + \dots + x_n^2$ and $\|x\|_\infty = \max_j |x_j|$.

The elementary properties of concentration functions are well studied (see, for instance, [2], [10], [15]). In particular, it is obvious that $Q(F, \mu) \le (1 + \lceil \mu/\lambda \rceil)\, Q(F, \lambda)$, for any $\mu, \lambda > 0$, where $\lceil x \rceil$ is the integer part of a number x. Hence,

$$Q(F, c\lambda) \asymp Q(F, \lambda) \tag{2}$$

and

$$\text{if } Q(F, \lambda) \ll B, \text{ then } Q(F, \mu) \ll B\,(1 + \mu/\lambda). \tag{3}$$

The problem of estimating the concentration function of weighted sums S_a under different conditions on the vector $a \in \mathbf{R}^n$ and distributions of summands has been studied in [9], [14], [17], [18], [19], [20]. Eliseeva and Zaitsev [4] have obtained some improvements of the results [9] and [18]. In this paper we formulate and prove similar refinements of a result of Vershynin [21].

The result of Vershynin [21], related to the Littlewood–Offord problem, is formulated as follows. Let $\log_+(x) = \max\{0, \log x\}$.

Proposition 1. *Let $X, X_1, \dots, X_n$ be i.i.d. random variables and $a = (a_1, \dots, a_n) \in \mathbf{R}^n$ with $\|a\| = 1$. Assume that there exist positive numbers τ, p, K, L, D such that $Q(\mathcal{L}(X), \tau) \le 1 - p$, $\mathbf{E}\,|X| \le K$, and*

$$\|\, ta - m\| \ge L\sqrt{\log_+(t/L)} \quad \textit{for all } m \in \mathbf{Z}^n \textit{ and } \; t \in (0, D]. \tag{4}$$

If $L^2 \ge 1/p$, then

$$Q\Big(F_a, \frac{1}{D}\Big) \le \frac{C\, L}{D}, \tag{5}$$

where the quantity C depends on τ, p, K only.

Corollary 1. *Let the conditions of Proposition 1 be satisfied. Then, for any $\varepsilon \ge 0$,*

$$Q(F_a, \varepsilon) \ll C\, L\,\Big(\varepsilon + \frac{1}{D}\Big). \tag{6}$$

It is clear that if

$$0 < D \le D(a) = D_L(a) = \inf\Big\{t > 0 : \operatorname{dist}(ta, \mathbf{Z}^n) < L\sqrt{\log_+(t/L)}\Big\}, \tag{7}$$

where

$$\operatorname{dist}(ta, \mathbf{Z}^n) = \min_{m \in \mathbf{Z}^n} \|\, ta - m\| = \sum_{k=1}^n \min_{m_k \in \mathbf{Z}} |\, ta_k - m_k|^2,$$

then condition (4) holds. In Vershynin [21] the quantity $D(a)$ is called the least common denominator of the vector $a \in \mathbf{R}^n$ (see also Rudelson and Vershynin [17] and [18] for similar definitions).

Note that for $|\,t| \le 1/2\,\|a\|_\infty$ we have

$$\big(\mathrm{dist}(ta, \mathbf{Z}^n)\big)^2 = \sum_{k=1}^{n} |\,ta_k|^2 = \|a\|^2 t^2 = t^2. \tag{8}$$

By definition, $D(a) > L$. Moreover, equality (8) implies that $D(a) \ge 1/2\,\|a\|_\infty$ (see Vershynin [21], Lemma 6.2).

Note that the statement of Corollary 1 with $D = D(a)$ is the version of the concentration result for the Littlewood–Offord problem as formulated in [21]. Proposition 1 seems to be more natural formulation which implies Corollary 1 using relations (3) and (7).

In the formulation of Proposition 1, w.l.o.g. we can replace assumption (4) by the following:

$$\|\,ta - m\| \ge f_L(t) \quad \text{ for all } m \in \mathbf{Z}^n \text{ and } \ t \in \Big[\frac{1}{2\,\|a\|_\infty}, D\Big], \tag{9}$$

where

$$f_L(t) = \begin{cases} t/6, & \text{for } 0 < t < eL, \\ L\sqrt{\log(t/L)}, & \text{for } t \ge eL. \end{cases} \tag{10}$$

Indeed, if $t \ge eL$, this follows from assumption (4). If $0 < t < eL$ and there exists an $m \in \mathbf{Z}^n$ such that $\|\,ta-m\| < t/6$, then, denoting $k = \lceil eL/t \rceil + 1$, we have $tk \ge eL$ and $\|\,tka - km\| < tk/6 \le 2eL/6 < L \le L\sqrt{\log_+(tk/L)}$. Since $km \in \mathbf{Z}^n$, we have $D \le D(a) \le tk \ll L$ and the required inequality (5) is a trivial consequence of $Q(F_a, 1/D) \le 1$.

Note that equality (8) justifies why the assumption $t \ge 1/2\,\|a\|_\infty$ in condition (9) is natural. For $0 < t < 1/2\,\|a\|_\infty$, inequality (9) is satisfied automatically.

It seems that the least common denominator $D^*(a)$ should be defined as

$$D^*(a) = \inf\Big\{t > 0 : \mathrm{dist}(ta, \mathbf{Z}^n) < f_L(t\|a\|)\Big\}. \tag{11}$$

This definition will be also used below in the case when $\|a\| \ne 1$. Obviously,

$$D^*(\lambda a) = D^*(a)/\lambda, \quad \text{ for any } \lambda > 0, \tag{12}$$

and equality (8) implies also that $D^*(a) \ge 1/2\,\|a\|_\infty$.

Now we formulate the main result of this paper.

Theorem 1. *Let $X, X_1, \dots, X_n$ be i.i.d. random variables. Let $a = (a_1, \dots, a_n) \in \mathbf{R}^n$ with $\|a\| = 1$. Assume that condition* (9) *is satisfied. If $L^2 \ge 1/M(1)$, where the quantity $M(1)$ is defined by formula* (1)*, then*

$$Q\Big(F_a, \frac{1}{D}\Big) \ll \frac{1}{D\sqrt{M(1)}}. \tag{13}$$

Let us reformulate Theorem 1 for arbitrary a, without assuming that $\|a\| = 1$.

Corollary 2. *Let the conditions of Theorem* 1 *be satisfied with condition* (9) *replaced by the condition*

$$\| \, ta - m\| \geq f_L(t\|a\|) \quad \textit{for all } m \in \mathbf{Z}^n \textit{ and } \; t \in \Big[\frac{1}{2\,\|a\|_\infty}, D\Big], \tag{14}$$

and without the assumption $\|a\| = 1$*. If* $L^2 \geq 1/M(1)$*, then*

$$Q\Big(F_a, \frac{1}{D}\Big) \ll \frac{1}{\|a\| D\sqrt{M(1)}}.$$

The proofs of our Theorem 1 and Corollary 2 are similar to the proof of the main results of Eliseeva and Zaitsev [4]. They are in some sense more natural than the proofs in Vershynin [21], since they do not use unnecessary assumptions like $\mathbf{E}\,|X| \leq K$. This is achieved by an application of relation (42). Our proof differs from the arguments used in [9], [18] and [21] since we rely on methods introduced by Esséen [6] (see the proof of Lemma 4 of Chapter II in [15]).

Now we reformulate Corollary 2 for the random variables X_k/τ, $\tau > 0$.

Corollary 3. *Let* $V_{a,\tau} = \mathcal{L}\big(\sum_{k=1}^n a_k X_k/\tau\big)$, $\tau > 0$*. Then, under the conditions of Corollary* 2 *with the condition* $L^2 \geq 1/M(1)$ *replaced by the condition* $L^2 \geq 1/M(\tau)$*, we have*

$$Q\Big(V_{a,\tau}, \frac{1}{D}\Big) = Q\Big(F_a, \frac{\tau}{D}\Big) \ll \frac{1}{\|a\| D\sqrt{M(\tau)}}. \tag{15}$$

In particular, if $\|a\| = 1$*, then*

$$Q\Big(F_a, \frac{\tau}{D}\Big) \ll \frac{1}{D\sqrt{M(\tau)}}. \tag{16}$$

For the proof of Corollary 3, it suffices to use relation (1).

It is evident that $M(\tau) \gg 1 - Q(G,\tau) \geq 1 - Q(F,\tau) \geq p$, where p is introduced in Proposition 1. Note that $M(\tau)$ can be essentially larger than p. For example, p may be equal to 0, while $M(\tau) > 0$ for any non-degenerate distribution $F = \mathcal{L}(X)$. Comparing the bounds (5) and (16), we see that the factor L is replaced by the factor $1/\sqrt{M(\tau)}$ which can be essentially smaller than L under the conditions of Corollary 3. Moreover, there is an unnecessary assumption $\mathbf{E}\,|X| \leq K$ in the formulation of Proposition 1. Finally, the dependence of constants on the distribution $\mathcal{L}(X)$ is stated explicitly, with absolute contants in the formulation. An improvement of Corollary 1 is given below in Theorem 2.

We recall now the well-known Kolmogorov–Rogozin inequality [16] (see [2], [10] and [15]).

Proposition 2. *Let $Y_1,\dots,Y_n$ be independent random variables with the distributions $W_k = \mathcal{L}(Y_k)$. Let $\lambda_1,\dots,\lambda_n$ be positive numbers such that $\lambda_k \le \lambda$, for $k=1,\dots,n$. Then*

$$Q\Big(\mathcal{L}\Big(\sum_{k=1}^n Y_k\Big),\lambda\Big) \ll \lambda \,\Big(\sum_{k=1}^n \lambda_k^2\,\big(1-Q(W_k,\lambda_k)\big)\Big)^{-1/2}. \tag{17}$$

Esséen [6] (see [15], Theorem 3 of Chapter III) has improved this result. He has shown that the following statement is true.

Proposition 3. *Under the conditions of Proposition 2 we have*

$$Q\Big(\mathcal{L}\Big(\sum_{k=1}^n Y_k\Big),\lambda\Big) \ll \lambda \,\Big(\sum_{k=1}^n \lambda_k^2\,M_k(\lambda_k)\Big)^{-1/2}, \tag{18}$$

where $M_k(\tau) = \mathbf{E}\,\min\big\{\widetilde{Y_k}^{\,2}/\tau^2, 1\big\}$.

Furthermore, improvements of (17) and (18) may be found in [1], [2], [3], [7], [8], [11], [12] and [13].

It is clear that Theorem 1 is related to Proposition 1 in a similar way as Esséen's inequality (18) is related to the Kolmogorov–Rogozin inequality (17).

If we consider the special case, where $D = 1/2\,\|a\|_\infty$, then no assumptions on the arithmetic structure of the vector a are made, and Corollary 3 implies the bound

$$Q(F_a, \tau\,\|a\|_\infty) \ll \frac{\|a\|_\infty}{\|a\|\sqrt{M(\tau)}}. \tag{19}$$

This result follows from Esséen's inequality (18) applied to the sum of non-identically distributed random variables $Y_k = a_k X_k$ with $\lambda_k = a_k\,\tau$, $\lambda = \|a\|_\infty\,\tau$. For $a_1 = a_2 = \cdots = a_n = n^{-1/2}$, inequality (19) turns into the well-known particular case of Proposition 3:

$$Q(F^{*n},\tau) \ll \frac{1}{\sqrt{n\,M(\tau)}}. \tag{20}$$

Inequality (20) implies as well the Kolmogorov–Rogozin inequality for i.i.d. random variables:

$$Q(F^{*n},\tau) \ll \frac{1}{\sqrt{n\,(1-Q(F,\tau))}}.$$

Inequality (19) can not yield bound of better order than $O(n^{-1/2})$, since the right-hand side of (19) is at least $n^{-1/2}$. The results stated above are more interesting if D is essentially larger than $1/2\,\|a\|_\infty$. In this case one can expect the estimates of smaller order than $O(n^{-1/2})$. Such estimates of $Q(F_a,\lambda)$ are required to study the distributions of eigenvalues of random matrices.

For $0 < D < 1/2\,\|a\|_\infty$, the inequality

$$Q\Big(F_a, \frac{\tau}{D}\Big) \ll \frac{1}{\|a\| D\sqrt{M(\tau)}} \tag{21}$$

holds asuuming the conditions of Corollary 3 too. In this case it follows from (3) and (19).

Under the conditions of Corollary 3, there exist many possibilities to represent a fixed ε as $\varepsilon = \tau/D$ for an appication of inequality (15). Therefore, for a fixed $\varepsilon = \tau/D$ we can try to minimize the right-hand side of inequality (15) choosing an optimal D. This is possible, and the optimal bound is given in the following Theorem 2.

Theorem 2. *Let the conditions of Corollary* 2 *be satisfied except the condition* $L^2 \geq 1/M(1)$. *Let* $L^2 > 1/P$, *where* $P = \mathbf{P}(\widetilde{X} \neq 0) = \lim_{\tau\to 0} M(\tau)$. *Then there exists a* τ_0 *such that* $L^2 = 1/M(\tau_0)$. *Moreover, the bound*

$$Q\big(F_a, \varepsilon\big) \ll \frac{1}{\|a\| D^*(a) \sqrt{M(\varepsilon\, D^*(a))}} \tag{22}$$

is valid for $0 < \varepsilon \leq \varepsilon_0 = \tau_0/D^*(a)$. *Furthermore, for* $\varepsilon \geq \varepsilon_0$, *the bound*

$$Q\big(F_a, \varepsilon\big) \ll \frac{\varepsilon L}{\varepsilon_0\, \|a\| D^*(a)} \tag{23}$$

holds.

In the statement of Theorem 2, the quantity ε can be arbitrarily small. If ε tends to zero, we obtain

$$Q(F_a, 0) \ll \frac{1}{\|a\| D^*(a) \sqrt{P}},$$

if $L^2 > 1/P$.

Theorem 2 follows easily from Corollary 3. Indeed, denoting $\varepsilon = \tau/D$, we can rewrite inequality (15) as

$$Q\big(F_a, \varepsilon\big) \ll \frac{1}{\|a\| D \sqrt{M(\varepsilon\, D)}}. \tag{24}$$

Inequality (24) holds if $L^2 \geq 1/M(\varepsilon\, D)$ and $0 < D \leq D^*(a)$. If $L^2 \geq 1/M(\varepsilon\, D^*(a))$, then the choice $D = D^*(a)$ is optimal in inequality (24) since

$$D^2 M(\varepsilon\, D) = \mathbf{E} \min\big\{\widetilde{X}^2/\varepsilon^2, D^2\big\}$$

is increasing when D increases. For the same reason, if $L^2 < 1/M(\varepsilon\, D^*(a))$, the optimal choice of D in inequality (24) is given by the solution $D_0(\varepsilon)$ of the equation $L^2 = 1/M(\varepsilon\, D)$. This solution exists and is unique if $L^2 > 1/P$, since the function $M(\tau)$ is continuous and strictly decreasing if $M(\tau) < P$. Moreover, $M(\tau) \to 0$ as $\tau \to \infty$. In this case inequality (24) turns into

$$Q\big(F_a, \varepsilon\big) \ll \frac{L}{\|a\| D_0(\varepsilon)}. \tag{25}$$

Moreover, choosing τ_0 to be the solution of the equation $L^2 = 1/M(\tau)$, we see that inequality (22) is valid for $0 < \varepsilon \le \varepsilon_0 = \tau_0/D^*(a)$. It is clear that $D_0(\varepsilon_0) = D^*(a)$. Furthermore, for $\varepsilon \ge \varepsilon_0$, we have

$$M(\varepsilon\, D_0(\varepsilon)) = M(\varepsilon_0\, D_0(\varepsilon_0)) = L^{-2}$$

and, hence, $\varepsilon\, D_0(\varepsilon) = \varepsilon_0\, D_0(\varepsilon_0)$. Therefore, for $\varepsilon \ge \varepsilon_0$, inequality (23) holds. Obviously, inequality (23) could be derived from (24) with $\varepsilon = \varepsilon_0$ by an application of inequality (3). On the other hand, for $0 < \varepsilon_1 < \varepsilon \le \varepsilon_0$, we could apply inequality (3) to inequality (22) and obtain the bound

$$Q\big(F_a, \varepsilon\big) \ll \frac{\varepsilon}{\varepsilon_1}\, Q\big(F_a, \varepsilon_1\big) \ll \frac{\varepsilon}{\varepsilon_1\, \|a\| D^*(a) \sqrt{M(\varepsilon_1\, D^*(a))}}. \tag{26}$$

However, inequality (26) is weaker than inequality (22) since, evidently,

$$\varepsilon^2 M(\varepsilon\, \mu) = \mathbf{E} \min\big\{\widetilde{X}^2/\mu^2, \varepsilon^2\big\} \ge \mathbf{E} \min\big\{\widetilde{X}^2/\mu^2, \varepsilon_1^2\big\} = \varepsilon_1^2\, M(\varepsilon_1\, \mu), \tag{27}$$

for any $\mu > 0$.

It is clear that Theorem 2 is an essential improvement of Corollary 1. In particular, in contrast with inequality (6) of Corollary 1, for small ε, the right-hand side of inequality (22) of Theorem 2 may be decreasing as ε decreases. Moreover, we have just shown that the application of inequality (3) would lead to a loss of precision. However, Corollary 1 could be derived from Proposition 1 with the help of inequality (3).

Consider a simple example. Let X be the random variable taking values 0 and 1 with probabilities

$$\mathbf{P}\{X = 1\} = 1 - \mathbf{P}\{X = 0\} = p > 0. \tag{28}$$

Then

$$\mathbf{P}\{\widetilde{X} = \pm 1\} = p(1-p), \quad \mathbf{P}\{\widetilde{X} = 0\} = 1 - 2\,p(1-p), \tag{29}$$

and the function $M(\tau)$ has the form

$$M(\tau) = \begin{cases} 2\,p(1-p), & \text{for } 0 < \tau < 1, \\ 2\,p(1-p)/\tau^2, & \text{for } \tau \ge 1. \end{cases} \tag{30}$$

Assume for simplicity that $\|a\| = 1$. If $L^2 > 1/2\,p(1-p)$, then $\tau_0 = L\sqrt{2\,p(1-p)}$ and, for $\varepsilon \ge \varepsilon_0 = L\sqrt{2\,p(1-p)}/D^*(a)$, we have the bound

$$Q\big(F_a, \varepsilon\big) \ll \frac{\varepsilon}{\sqrt{p(1-p)}}. \tag{31}$$

The same bound (31) follows from inequality (22) of Theorem 2 for $1/D^*(a) \le \varepsilon \le \varepsilon_0$. For $0 < \varepsilon \le 1/D^*(a)$, inequality (22) implies the bound

$$Q\big(F_a, \varepsilon\big) \ll \frac{1}{D^*(a)\sqrt{p(1-p)}}. \tag{32}$$

Thus,

$$Q\big(F_a,\varepsilon\big) \ll \min\Big\{\frac{1}{\sqrt{p(1-p)}}\Big(\varepsilon+\frac{1}{D^*(a)}\Big),\,1\Big\},\quad \text{for all } \varepsilon\geq 0. \tag{33}$$

Inequality (33) is stronger than (6) since the factor L disappears completely. Moreover, this inequality cannot be improved. Consider, for instance, $a=(s^{-1/2},\dots,s^{-1/2},0,\dots,0)$ with the first $s\leq n$ coordinates equal to $s^{-1/2}$ and the last $n-s$ coordinates equal to zero. In this case $D^*(a)\asymp s^{1/2}$, the random variable $s^{1/2}S_a$ has binomial distribution with parameters s and p, and it is well-known that

$$Q\big(F_a,\varepsilon\big) \gg \min\Big\{\frac{1}{\sqrt{p(1-p)}}\Big(\varepsilon+\frac{1}{\sqrt{s}}\Big),\,1\Big\},\quad \text{for all } \varepsilon\geq 0. \tag{34}$$

Comparing the bounds (33) and (34), we see that Theorem 2 provides the optimal order of $Q\big(F_a,\varepsilon\big)$ for all possible values of ε. Moreover, the involved constants are absolute.

For the sake of completeness, we give below a short proof of inequality (34). It is easy to see that $\operatorname{Var}\big(S_a\big)=p(1-p)$. Therefore, by Chebyshev's inequality,

$$\mathbf{P}\big\{|S_a-\mathbf{E}\,S_a|<2\sqrt{p(1-p)}\big\}\geq 3/4. \tag{35}$$

The random variable S_a takes values which are multiples of $s^{-1/2}$. Therefore, if $s\,p(1-p)\leq 1$, then inequality (35) implies that $Q\big(F_a,0\big)\asymp 1$ and inequality (34) is trivially valid.

Assume now $s\,p(1-p)>1$. If $0<\varepsilon\leq 4\sqrt{p(1-p)}$, then, using (3) and (35), we obtain

$$3/4\leq Q\big(F_a,4\sqrt{p(1-p)}\big)\ll \varepsilon^{-1}\sqrt{p(1-p)}\,Q\big(F_a,\varepsilon\big), \tag{36}$$

and, hence,

$$Q\big(F_a,\varepsilon\big)\gg\frac{\varepsilon}{\sqrt{p(1-p)}}. \tag{37}$$

It is clear that (2), (3) and (37) imply that $Q\big(F_a,\varepsilon\big)\asymp 1$, for $\varepsilon\geq 4\sqrt{p(1-p)}$. Applying inequality (37) for $\varepsilon=s^{-1/2}$ and using the lattice structure of the support of distribution F_a, we conclude that, for $0\leq\varepsilon<s^{-1/2}$,

$$Q\big(F_a,\varepsilon\big)\geq Q\big(F_a,0\big)\gg\frac{1}{\sqrt{s\,p(1-p)}}. \tag{38}$$

Thus, inequalities (2), (3), (37) and (38) imply (34).

The quantity $\tau_0=\varepsilon_0\,D^*(a)$ (which is the solution of the equation $L^2=1/M(\tau)$) may be interpreted as a quantity depending on L and on the distribution $\mathcal{L}(X)$. Moreover, comparing the bounds (6) and (23) for relatively large values of ε, we see that $\tau_0\to\infty$ as $L\to\infty$. Therefore, the factor L/τ_0 is much smaller than L for large values of L. In particular, in the example above we have $\tau_0=L\sqrt{2\,p(1-p)}$.

Another example would be a symmetric stable distribution with parameter α, $0<\alpha<2$. In this case the characteristic function $\widehat{F}(t)=\mathbf{E}\,\exp(itX)$ has the form $\widehat{F}(t)=\exp(-|t|^\alpha)$. It could be shown that then τ_0 behaves as $L^{2/\alpha}$ as $L\to\infty$.

Inequality (31) can be rewritten in the form

$$Q\big(F_a, \varepsilon\big) \ll \frac{\varepsilon}{\sigma}, \quad \text{for } \varepsilon \geq \varepsilon_0, \tag{39}$$

where $\sigma^2 = \operatorname{Var}(X)$. It is clear that a similar situation occurs for any random variable X with finite variance. In particular, inequality (31) is obviously satisfied for all $\varepsilon \geq 0$, if $\|a\| = 1$ and X has a Gaussian distribution with Var $(X) = \sigma^2$.

2. Proofs

We will use the classical Esséen inequalities ([5], see also [10] and [15]):

$$Q(F, \lambda) \ll \lambda \int_0^{\lambda^{-1}} |\widehat{F}(t)|\, dt, \quad \lambda > 0, \tag{40}$$

where $\widehat{F}(t)$ is the characteristic function of the corresponding random variable. In the general case $Q(F, \lambda)$ cannot be estimated from below by the right hand side of inequality (40). However, if we assume additionally that the distribution F is symmetric and its characterictic function is non-negative for all $t \in \mathbf{R}$, then we have the lower bound:

$$Q(F, \lambda) \gg \lambda \int_0^{\lambda^{-1}} \widehat{F}(t)\, dt \tag{41}$$

and, therefore,

$$Q(F, \lambda) \asymp \lambda \int_0^{\lambda^{-1}} \widehat{F}(t)\, dt \tag{42}$$

(see [2], Lemma 1.5 of Chapter II). The use of relation (42) allowed us to simplify the arguments of Friedland and Sodin [9], Rudelson and Vershynin [18] and Vershynin [21] which were applied to the Littlewood–Offord problem (see also Eliseeva and Zaitsev [4]).

Proof of Theorem 1. Let r be a fixed number satisfying $1 < r \leq \sqrt{2}$. Represent the distribution $G = \mathcal{L}(\widetilde{X})$ as a mixture $G = qE + \sum_{j=0}^{\infty} p_j G_j$, where $q = \mathbf{P}(\widetilde{X} = 0)$, $p_j = \mathbf{P}(\widetilde{X} \in A_j)$, $j = 0, 1, 2, \ldots$, $A_0 = \{x : |x| > 1\}$, $A_j = \{x : r^{-j} < |x| \leq r^{-j+1}\}$, E is probability measure concentrated in zero, G_j are probability measures defined for $p_j > 0$ by the formula $G_j\{X\} = \dfrac{1}{p_j} G\{X \bigcap A_j\}$, for any Borel set X. In fact, G_j is the conditional distribution of $\widetilde{X}$ provided that $\widetilde{X} \in A_j$. If $p_j = 0$, then we can take as G_j arbitrary measures.

For $z \in \mathbf{R}$, $\gamma > 0$, introduce the distribution $H_{z,\gamma}$, with the characteristic function

$$\widehat{H}_{z,\gamma}(t) = \exp\Big(-\frac{\gamma}{2} \sum_{k=1}^{n} \big(1 - \cos(2a_k z t)\big)\Big). \tag{43}$$

It is clear that $H_{z,\gamma}$ is a symmetric infinitely divisible distribution. Therefore, its characteristic function is positive for all $t \in \mathbf{R}$.

For the characteristic function $\widehat{F}(t) = \mathbf{E}\exp(itX)$, we have

$$|\widehat{F}(t)|^2 = \mathbf{E}\exp(it\widetilde{X}) = \mathbf{E}\cos(t\widetilde{X}),$$

where $\widetilde{X} = X_1 - X_2$ is the corresponding symmetrized random variable. Hence,

$$|\widehat{F}(t)| \le \exp\Big(-\frac{1}{2}\big(1-|\widehat{F}(t)|^2\big)\Big) = \exp\Big(-\frac{1}{2}\,\mathbf{E}\big(1-\cos(t\widetilde{X})\big)\Big). \tag{44}$$

According to (40) and (44), we have

$$\begin{aligned} Q(F_a, 1/D) &\ll \frac{1}{D}\int_0^D |\widehat{F_a}(t)|\,dt \\ &\ll \frac{1}{D}\int_0^D \exp\Big(-\frac{1}{2}\sum_{k=1}^n \mathbf{E}\big(1-\cos(2a_k t\widetilde{X})\big)\Big)\,dt = I. \end{aligned} \tag{45}$$

It is evident that

$$\begin{aligned} \sum_{k=1}^n \mathbf{E}\big(1-\cos(2a_k t\widetilde{X})\big) &= \sum_{k=1}^n \int_{-\infty}^{\infty} \big(1-\cos(2a_k tx)\big)\,G\{dx\} \\ &= \sum_{k=1}^n \sum_{j=0}^{\infty} \int_{-\infty}^{\infty} \big(1-\cos(2a_k tx)\big)\,p_j\,G_j\{dx\} \\ &= \sum_{j=0}^{\infty} \sum_{k=1}^n \int_{-\infty}^{\infty} \big(1-\cos(2a_k tx)\big)\,p_j\,G_j\{dx\}. \end{aligned}$$

We denote $\beta_j = r^{-2j}p_j$, $\beta = \sum_{j=0}^{\infty}\beta_j$, $\mu_j = \beta_j/\beta$, $j = 0, 1, 2, \ldots$. It is clear that $\sum_{j=0}^{\infty}\mu_j = 1$ and $p_j/\mu_j = r^{2j}\beta$ (for $p_j > 0$).

Let us estimate the quantity β:

$$\begin{aligned} \beta = \sum_{j=0}^{\infty}\beta_j &= \sum_{j=0}^{\infty} r^{-2j}p_j = \mathbf{P}\big\{|\widetilde{X}| > 1\big\} + \sum_{j=1}^{\infty} r^{-2j}\,\mathbf{P}\big\{r^{-j} < |\widetilde{X}| \le r^{-j+1}\big\} \\ &\ge \int\limits_{|x|>1} G\{dx\} + \sum_{j=1}^{\infty} \int\limits_{r^{-j}<|x|\le r^{-j+1}} \frac{x^2}{r^2}\,G\{dx\} \\ &\ge \frac{1}{r^2}\int\limits_{|x|>1} G\{dx\} + \frac{1}{r^2}\int\limits_{|x|\le 1} x^2\,G\{dx\} = \frac{1}{r^2}M(1). \end{aligned}$$

Since $1 < r \le \sqrt{2}$, this implies

$$\beta \ge \frac{1}{2}M(1). \tag{46}$$

Condition $L^2 \geq 1/M(1)$ gives the bound

$$L^2\beta \geq \frac{1}{2}. \tag{47}$$

We now proceed similarly to the proof of a result of Esséen [6] (see [15], Lemma 4 of Chapter II). Using the Hölder inequality, it is easy to see that

$$I \leq \prod_{j=0}^{\infty} I_j^{\mu_j}, \tag{48}$$

where

$$\begin{aligned} I_j &= \frac{1}{D}\int_0^D \exp\Big(-\frac{p_j}{2\,\mu_j}\sum_{k=1}^n \int_{-\infty}^{\infty}\big(1-\cos(2a_k tx)\big)\,G_j\{dx\}\Big)\,dt \\ &= \frac{1}{D}\int_0^D \exp\Big(-\frac{1}{2}\,r^{2j}\beta\sum_{k=1}^n \int_{A_j}\big(1-\cos(2a_k tx)\big)\,G_j\{dx\}\Big)\,dt \end{aligned}$$

if $p_j > 0$, and $I_j = 1$ if $p_j = 0$.

Applying Jensen's inequality to the exponential in the integral (see [15], p. 49)), we obtain

$$\begin{aligned} I_j &\leq \frac{1}{D}\int_0^D \int_{A_j} \exp\Big(-\frac{1}{2}\,r^{2j}\beta\,\sum_{k=1}^n \big(1-\cos(2a_k tx)\big)\Big)\,G_j\{dx\}\,dt \\ &= \frac{1}{D}\int_{A_j}\int_0^D \exp\Big(-\frac{1}{2}\,r^{2j}\beta\,\sum_{k=1}^n \big(1-\cos(2a_k tx)\big)\Big)\,dt\,G_j\{dx\} \\ &\leq \sup_{z\in A_j}\frac{1}{D}\int_0^D \widehat{H}_{z,1}^{r^{2j}\beta}(t)\,dt. \end{aligned} \tag{49}$$

Let us estimate the characterictic function $\widehat{H}_{\pi,1}(t)$ for $|\,t| \leq D$. We can proceed in the same way as the authors of [9], [18] and [21]. It is evident that $1-\cos x \geq 2x^2/\pi^2$, for $|x| \leq \pi$. For arbitrary x, this implies that $1-\cos x \geq 2\,\pi^{-2}\min_{m\in\mathbf{Z}}|\,x-2\pi m|^2$. Substituting this inequality into (43), we obtain

$$\begin{aligned} \widehat{H}_{\pi,1}(t) &\leq \exp\Big(-\frac{1}{\pi^2}\sum_{k=1}^n \min_{m_k\in\mathbf{Z}}\big|2\pi t a_k - 2\pi m_k\big|^2\Big) \\ &= \exp\Big(-4\sum_{k=1}^n \min_{m_k\in\mathbf{Z}}|\,t a_k - m_k|^2\Big) \\ &= \exp\big(-4\,\big(\mathrm{dist}(ta,\mathbf{Z}^n)\big)^2\big). \end{aligned} \tag{50}$$

Using (8), wee see that, for $|\,t| \leq 1/2\,\|a\|_\infty$, inequality (50) turns into

$$\widehat{H}_{\pi,1}(t) \leq \exp(-4\,t^2). \tag{51}$$

Now we can use relations (9), (50) and (51) to estimate the integrals I_j. First we consider the case $j = 1, 2, \ldots$. Note that the characteristic functions $\widehat{H}_{z,\gamma}(t)$ satisfy the equalities

$$\widehat{H}_{z,\gamma}(t) = \widehat{H}_{y,\gamma}\big(zt/y\big) \quad \text{and} \quad \widehat{H}_{z,\gamma}(t) = \widehat{H}^{\gamma}_{z,1}(t). \tag{52}$$

The first equality (52) implies that

$$\text{if} \quad H_{z,\gamma} = \mathcal{L}(\xi), \quad \text{then} \quad H_{y,\gamma} = \mathcal{L}(y\,\xi/z). \tag{53}$$

For $z \in A_j$ we have $r^{-j} < |z| \leq r^{-j+1} < \pi$. Hence, for $|\,t| \leq D$, we have $|zt/\pi| < D$. Therefore, using the properties (52) with $y = \pi$ and aforementioned estimates (9), (50) and (51), we obtain, for $z \in A_j$ and for $z = \pi$,

$$\begin{aligned} \widehat{H}_{z,1}(t) &\leq \exp\big(-4\, f_L^2(zt/\pi)\big) \\ &= \begin{cases} \exp\big(-(zt/\pi)^2/9\big), & \text{for } 0 < t \leq eL\pi/z, \\ \exp\big(-4\,L^2\,\log(zt/L\pi)\big), & \text{for } t > eL\pi/z. \end{cases} \end{aligned}$$

and, hence,

$$\sup_{z\in A_j} \int\limits_0^D \widehat{H}^{r^{2j}\beta}_{z,1}(t)\,dt \leq \int\limits_0^D \exp\big(-t^2\beta/9\pi^2\big)\,dt + \int\limits_{r^{j-1}L\pi e}^{\infty} \Big(\frac{r^j L\pi}{t}\Big)^{4\,r^{2j}\beta L^2} dt \ll \frac{1}{\sqrt{\beta}}. \tag{54}$$

In the last inequality we used inequality (47).

Consider now the case $j = 0$. The relation (53) yields, for $z > 0$, $\gamma > 0$,

$$Q(H_{z,\gamma}, 1/D) = Q\big(H_{1,\gamma}, 1/Dz\big). \tag{55}$$

Thus, according to (2), (42), (52) and (55), we obtain

$$\begin{aligned} \sup_{z\in A_0} \frac{1}{D}\int_0^D \widehat{H}^{\beta}_{z,1}(t)\,dt &= \sup_{z>1} \frac{1}{D}\int_0^D \widehat{H}_{z,\beta}(t)\,dt \asymp \sup_{z>1}\ Q(H_{z,\beta}, 1/D) \\ &= \sup_{z>1}\ Q\big(H_{1,\beta}, 1/Dz\big) \leq Q(H_{1,\beta}, 1/D) \\ &\asymp Q\big(H_{1,\beta}, 1/D\pi\big) = Q(H_{\pi,\beta}, 1/D) \\ &\asymp \frac{1}{D}\int_0^D \widehat{H}_{\pi,\beta}(t)\,dt = \frac{1}{D}\int_0^D \widehat{H}^{\beta}_{\pi,1}(t)\,dt. \end{aligned} \tag{56}$$

Using the bounds (9), (50) and (51) for the characteristic function $\widehat{H}_{\pi,1}(t)$ and taking into account inequality (47), we have:

$$\int_0^D \widehat{H}^{\beta}_{\pi,1}(t)\,dt \leq \int_0^D \exp(-t^2\beta/9)\,dt + \int_{Le}^{\infty} \Big(\frac{L}{t}\Big)^{4\beta L^2} dt \ll \frac{1}{\sqrt{\beta}}. \tag{57}$$

According to (49), (54), (56) and (57), we obtained the same estimate

$$I_j \ll \frac{1}{D\sqrt{\beta}} \tag{58}$$

for all integrals I_j with $p_j \neq 0$. In view of $\sum_{j=0}^{\infty} \mu_j = 1$, from (48) and (58) it follows that

$$I \leq \prod_{j=0}^{\infty} I_j^{\mu_j} \ll \frac{1}{D\sqrt{\beta}}. \tag{59}$$

Using (45), (46) and (59), we complete the proof. □

Now we will deduce Corollary 2 from Theorem 1.

Proof of Corollary 2. We denote $b = a/\|a\| \in \mathbf{R}^n$. Then the equality $Q(F_a, \lambda) = Q(F_b, \lambda/\|a\|)$, for all $\lambda \geq 0$, holds. The vector b satisfies the conditions of Theorem 1 which hold for the vector a when replacing D by $D\|a\|$. Indeed, $\|ub - m\| \geq f_L(u)$ for $u \in \Big[\dfrac{1}{2\,\|b\|_\infty}, D\|a\|\Big]$ and for all $m \in \mathbf{Z}^n$. This follows from condition (9) of Theorem 1, if we denote $u = t\|a\|$. It remains to apply Theorem 1 to the vector b. □

References

[1] Arak T.V. On the convergence rate in Kolmogorov's uniform limit theorem. I. Theory Probab. Appl., 1981, v. 26, no. 2, 225–245.

[2] Arak T.V., Zaitsev A.Yu. Uniform limit theorems for sums of independent random variavles. Proc. Steklov Inst. Math., 1988, v. 174.

[3] Bretagnolle J. Sur l'inégalité de concentration de Doeblin–Lévy, Rogozin–Kesten. In: Parametric and semiparametric models with applications to reliability, survival analysis, and quality of life, Stat. Ind. Technol., Birkhäuser Boston, Boston, MA, 2004, pp. 533–551.

[4] Eliseeva Yu.S., Zaitsev A.Yu. Estimates for the concentration functions of weighted sums of independent random variables. arXiv:1203.5520, 2012 (submitted to Theory Probab. Appl.).

[5] Esseen C.G. On the Kolmogorov–Rogozin inequality for the concentration function. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 1966, v. 5, 210–216.

[6] Esseen C.G. On the concentration function of a sum of independent random variables. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 1968, v. 9, 290–308.

[7] Götze F., Zaitsev A.Yu. Estimates for the rapid decay of concentration functions of n-fold convolutions. J. Theoret. Probab., 1998, v. 11, no. 3, 715–731.

[8] Götze F., Zaitsev A.Yu. A multiplicative inequality for concentration functions of n-fold convolutions. In: High dimensional probability, II (Seattle, WA, 1999), Progr. Probab., v. 47, Birkhäuser Boston, Boston, MA, 2000, pp. 39–47.

[9] Friedland O., Sodin S. Bounds on the concentration function in terms of Diophantine approximation. C. R. Math. Acad. Sci. Paris, 2007, v. 345, no. 9, 513–518.

[10] Hengartner W., Theodorescu R. Concentration function. Academic Press, New York, 1973.

[11] Kesten H. A sharper form of the Doeblin–Levy–Kolmogorov–Rogozin inequality for concentration functions. Math. Scand., 1969, v. 25, 133–144.

[12] Miroshnikov A.L., Rogozin B.A. Inequalities for the concentration functions. Theory Probab. Appl., 1980, v. 25, no. 4, 178–183.

[13] Nagaev S.V., Hodzhabagyan S.S. On the estimate for the concentration function of sums of independent random variables. Theory Probab. Appl., 1996, v. 41, no. 3, 655–665.

[14] Nguyen Hoi, Vu Van. Optimal inverse Littlewood–Offord theorems. Adv. Math., 2011, v. 226, no. 6. 5298–5319.

[15] Petrov V.V. Sums of independent random variables. Moscow, Nauka, 1972.

[16] Rogozin B.A. On the increase of dispersion of sums of independent random variables. Theory Probab. Appl., 1961, v. 6, no. 1, 106108.
[17] Rudelson M., Vershynin R. The Littlewood–Offord problem and invertibility of random matrices. Adv. Math., 2008, v. 218, no. 2, 600–633.
[18] Rudelson M., Vershynin R. Smallest singular value of a random rectangular matrix. Comm. Pure Appl. Math., 2009, v. 62, no. 12, 1707–1739.
[19] Tao T., Vu Van. Inverse Littlewood–Offord theorems and the condition number of random discrete matrices. Ann. of Math. (2), 2009. v. 169, no. 2, 595–632.
[20] Tao T., Vu Van. From the Littlewood–Offord problem to the circular law: universality of the spectral distribution of random matrices. Bull. Amer. Math. Soc. (N.S.), 2009, v. 46, no. 3, 377–396.
[21] Vershynin R. Invertibility of symmetric random matrices. arXiv:1102.0300, 2011.

E-mail address: pochta106@yandex.ru

St. Petersburg State University

E-mail address: goetze@math.uni-bielefeld.de

Fakultät für Mathematik,
Universität Bielefeld, Postfach 100131,
D-33501 Bielefeld, Germany

E-mail address: zaitsev@pdmi.ras.ru

St. Petersburg Department of Steklov Mathematical Institute
Fontanka 27, St. Petersburg 191023, Russia
and St. Petersburg State University